International Journal of Computer & Information Technologies (IJOCIT)
www.ijocit.ir & www.ijocit.org
ISSN= 2345-3877

Modeling the Competitive Facility Location Problem in a Symmetric Arena

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Abstract: In this paper, we consider the competitive facility location problem as a version of n-round Manhatan-metric Voronoi game with two players, where the distance measure is the Manhattan metric. Players alternate placing points, one at a time, into the playing arena that is a symmetric polygon, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule under the Manhattan distance, and the player whose points control the larger area wins. We study a winning strategy for the second player in a special version of the game.

Keywords: Computational geometry; Voronoi diagram; Voronoi game; Game theory; Competitive facility location.

1. Introduction

Suppose that two competing market players alternate placing their facilities one at a time, in a city. Let us assume that customers are equally distributed and that each customer shops at the

International Journal of Computer & Information Technologies (IJOCIT)
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August 23, 2013 Volume 1, Issue 1

market closest to her residence. Also assume that distance between two points is an orthogonal distance like perpendicular city streets. The Voronoi Game is a simple geometric model for the competitive facility location. Competitive facility location studies the placement of sites by competing market players. The geometric concepts are combined with game theory arguments to study if there exists any winning strategy. The Voronoi Game is played by two players, White and Black, who place a specified number, n, of facilities in a region Q. They alternate placing their facilities one at a time, with White going first. No point that has been occupied can be changed or reused by either player. After all $2n$ facilities have been placed, their decisions are evaluated by considering the Voronoi diagram of the $2n$ points in Q. At the end of the game, the Voronoi diagram of these $2n$ points is constructed; each player wins the total area of all cells belonging to points in his or her set. The player with the larger total area wins.

The most natural Voronoi game is played in a two-dimensional arena Q using the Euclidean metric. But in numerous applications the Euclidean metric does not provide an appropriate way of measuring distance. Up to now, nobody seems to know how to win this game, even for very restricted regions Q, unless the game is reduced to a single round [2, 3]. Ahn et al. [1] showed that for a one-dimensional arena, i.e., a line segment $[0, 2n]$, Black can win the n-round game, in which each player places a single point in each turn; however, White can keep Blacks winning margin arbitrarily small. This differs from the one-round game, in which both players get a single turn with n points each: Here, White can force a win by playing the odd integer points {1, 3, …, 2n-1}, again, the losing player can make the margin as small as he wishes. The used strategy focuses on key points. The question raised in the end of that paper is whether a similar notion can be extended to the two-dimensional scenario. Cheong et al. [2] showed that the two- or higher-dimensional scenario differs significantly: For sufficiently large $n \geq n_0$ and a square playing surface Q, the second player has a winning strategy that guarantees at least a fixed fraction of $1/2 + \alpha$ of the total area. Their proof uses a clever combination of probabilistic arguments to show that Black will do well by playing a random point.

In [3], Fekete and Meijer consider the one-round Voronoi game, in a rectangular area of aspect ratio $p \leq 1$. They showed that Black has a winning strategy for $n \geq 3$ and $p > \sqrt{2}/n$, and for $n = 2$ and $p > \sqrt{3}/2$. White wins in all remaining cases, i.e., for $n \geq 3$ and $p \leq \sqrt{2}/n$, for $n = 2$ and $p \leq \sqrt{3}/2$, and for $n = 1$. They also discuss complexity aspects of the game on more general boards, by proving that for a polygon with holes, it is NP-hard to maximize the area Black can win against a given set of points by White.

In this paper we present a strategy for winning a special version of two-dimensional n-round game, where the arena is symmetric, and it does not contain the center of symmetry and the distance measure is the Manhattan norm.

2. Preliminaries

We start with a definition of the n-round Voronoi game. There are two players, White and Black, each having n points to play. The players alternate placing points on a playing board Q which is symmetric. White starts the game, placing the first point, Black the second point, White the third point, etc., until all $2n$ points are played. We assume that points cannot lie upon each other.

Let $\mathbf{W} = \{w_1, w_2, \ldots, w_n\}$ be the set of white points at the end of the game and $\mathbf{B} = \{b1, b2, \ldots, b_n\}$ be the set of black points. At the end of the game, a Voronoi diagram of $\mathbf{W} \cup \mathbf{B}$ is constructed; each player wins the total area of all cells belonging to points in his or her set. The player with the larger total area wins. This game is called Manhattan-Voronoi game when we use the Manhattan distance instead of Euclidean. Let $\mathbf{q} = (x_q, y_q)$ and $\mathbf{r} = (x_r, y_r)$ be two points in the plane, the Manhattan distance of $\mathbf{q}$ and $\mathbf{r}$ is defined by the equation

$$L_1(q,r) = |x_q - x_r| + |y_q - y_r|$$

In Manhattan Voronoi diagram, the bisector is defined with Manhattan metric. Note that if two points are the diagonal vertices of an aligned square then their bisector is no longer a curve: it consists of two quarter-planes connected by a line segments. So we assume that points are in general position, i.e., there are no such pairs of points. The bisector with horizontal parts is called pseudo-horizontal and the bisector with vertical parts is called pseudo-vertical. The Voronoi cells in Manhattan Voronoi diagram is not necessarily convex, but it always star-shaped with respect to its site. Every edge of a cell consists of at most three straight lines that are parallel to the x-axis, y-axis, or diagonal lines within angle **л/4** or **3л/4**. The cell of site **q** is infinite if **q** is on the boundary of the convex hull of sites, but not conversely.

See figure 1.

The center of symmetry in region ***Q*** is denoted by ***O***. All distances are measured according to the Manhattan norm. Generally, for a set of points ***S***, the Manhattan Voronoi diagram of ***S*** is denoted by ***MV(S)***. If ***p*** ϵ ***S***, then $\mathbf{C_{(p)}}$ denotes the Voronoi cell of p in **MV(S)** and $|\mathbf{C_{(p)}}|$ denotes the area of $\mathbf{C_{(p)}}$.

If ***p*** = ***(x, y)***, the Voronoi cell of ***p*** is also denoted by ***C(x, y)***.

Lemma 1: A Voronoi cell in Manhattan-metric Voronoi diagram has no 45^{o} angel by the general position assumption.

Let ***p*** and ***q*** be two adjacent sites. If the bisector of line segment ***pq*** is pseudo-horizontal (or vertical), the common edge between them is called pseudo-horizontal (or vertical). Let ***e*** and e' be two adjacent edges of $C_{(p)}$. There are four cases:

1. ***e*** is horizontal and e' is vertical,
2. ***e*** is pseudo-horizontal and e' is vertical , or ***e*** is horizontal and e' is pseudo-vertical,
3. ***e*** is pseudo-horizontal and e' is pseudo-vertical,

4. ***e*** is pseudo-horizontal and ***e′*** is (pseudo-)horizontal, or ***e*** is pseudo-vertical and ***e′*** is pseudo-(vertical).

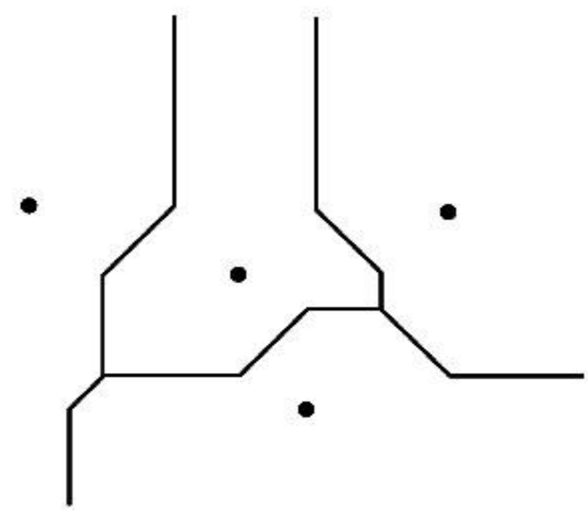

Figure 1: *Manhattan-Voronoi diagram*

In case 1, the intersection is called 90^o -intersection (figure 2a) or 270^o-intersection (figure 2b). In case 2, if the horizontal part of e intersects e0, the intersection is 90-intersection (figure 2c) or 270^o-intersection (figure 2d). If the diagonal part of e intersects ***e′*** , the intersection is called pseudo- 90^o -intersection (figure 2e) or pseudo- 270^o -intersection (figure 2f). The intersection is similarly defined for horizontal ***e*** and pseudo-vertical ***e′*** in case 2. In case 3, if the horizontal part of ***e*** intersects a vertical part of ***e′***, the intersection is 90^o -intersection (figure 2g) or 270^o -intersection (figure 2h), otherwise it is a pseudo - 90^o -intersection (figure 2e) or pseudo-270^o -intersection (figure 2f). In case 4, the intersection is called 0^o -intersection (figure 2i).

Let **p** be a site in **S** with finite region. Let us assume that the origin of coordinate system is at **p** and call the four areas counterclockwise by area ***I, II, III*** and ***IV*** and $\{p_1, p_2, \ldots, p_m\} \subset S$ is the neighbors of **p** in Manhattan-Voronoi

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diagram which are sorted around ***p*** conterclockwisely in order to their angle with positive direction of **X**-axis.

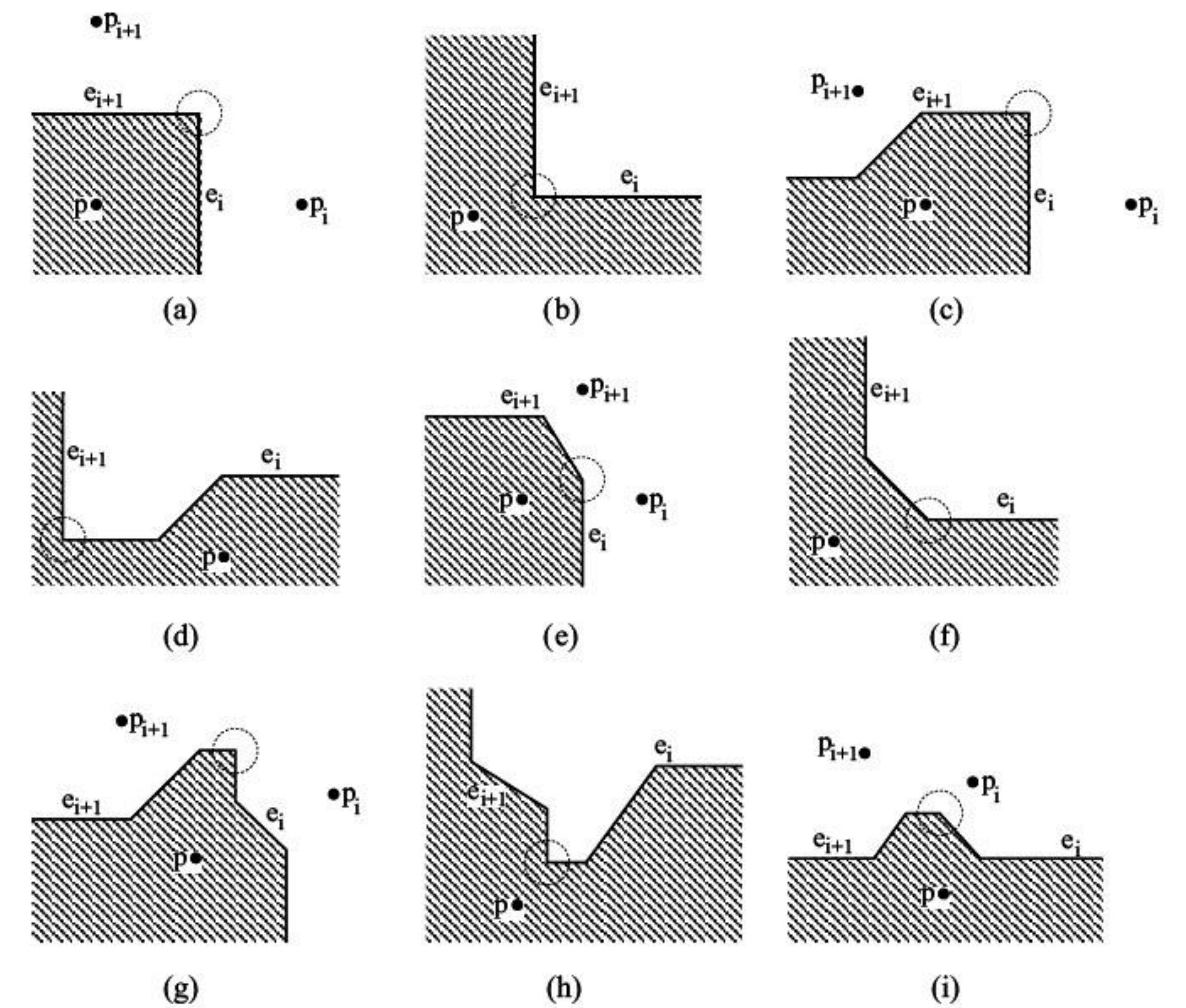

Figure 2: *(a), (c) and (g) are* 90°*-intersections, (b), (d) and (h) are* 270°*-intersections, (e) is pseudo*−90°*-intersection, (h) is pseudo* − 270°*-intersection and (i) is* 0°*-intersection.*

The intersection between two polygons $\boldsymbol{C_{(p)}}$ and $\boldsymbol{C_{(p_i)}}$ is called Voronoi edge and denoted by $\boldsymbol{e_i}$. Note that $\boldsymbol{e_i}$ consists of at most three straight lines that are parallel to the ***x-axis***, ***y-axis***, or diagonal lines within angle л/4 or 3л/4. The bisector of line segment $\boldsymbol{pp_i}$ is denoted by $\boldsymbol{b_i}$. If $\boldsymbol{b_i}$

is pseudo-horizontal (or vertical), e_i is called pseudo-horizontal (or vertical).

We divide the plane into eight areas by lines $x = 0$, $y = 0$, $y = x$ and $y = -x$. We denote the obtained open areas by $A_1, A_2, \ldots, A_8$ as shown in figure 3. The horizontal half-lines are denoted by H_1 and H_2 and vertical half-lines are denoted by V_1 and V_2. Note that if p_i is inside areas A_2, A_3, A_6 and A_7, then b_i is pseudo-horizontal, if p_i is inside areas $A1, A4, A5$ and $A8$, then b_i is pseudo-vertical, if pi is on V_1 and V_2, then b_i is horizontal; if p_i is on H_1 and H_2, then b_i is vertical. By the general position assumption, p_i does not lie on diagonal lines.

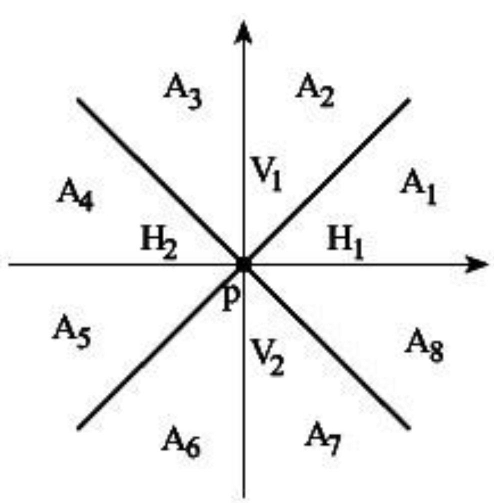

Figure 3: *The areas* $A_1, A_2, \ldots, A_8$

Lemma 2: A Voronoi cell in Manhattan-Voronoi diagram does not have any 270° and pseudo-270° -intersections.

Proof: We assume for a contradiction that $C_{(P)}$ has a 270° -intersection which is made by edges e_i and e_{i+1}. Without less of generality, suppose that e_{i+1} is vertical (or pseudo-vertical) and e_i is horizontal (or pseudo-horizontal). First we note that in counterclockwise order if p_i is before p_{i+1}, then e_i will be before e_{i+1}. As we mentioned, the $MV(S)$ regions are star-shaped, and p can visit the whole boundary of $C_{(p)}$. Then p should be in region R which is shown in figure 4.p_i should lie on top of e_i and inside area $A_2 \cup A_3$ and p_{i+1} should be at right side of e_{i+1} and inside area A_8 [A_1. Whereas, in counterclockwise order the sites inside $A_8 \cup A_1$ are before the sites inside $A_2 \cup A_3$. That is a contradiction. Then $C_{(p)}$ does not have any 270° and similarly pseudo-270 intersections.

Lemma 3: A bounded Voronoi cell in Manhattan-Voronoi diagram has exactly four 90o or pseudo-90° -intersections. Moreover, there is exactly one 90° or pseudo-90° -intersection inside each area ***I, II, III*** or ***IV***.

Proof: Consider four areas $A_2 \cup A_3 \cup V_1$, $A_4 \cup A_5 \cup H_2$, $A_6 \cup A_7 \cup V_2$ and $A_8 \cup A_1 \cup H_1$. We know that, for the sites inside $A_8 \cup A_1 \cup H_1$ and $A_4 \cup A_5 \cup H_2$ the edges of $C_{(p)}$ are vertical or pseudo-vertical and for the sites inside $A_2 \cup A_3 \cup V_1$ and $A_6 \cup A_7 \cup V_2$ the edges of $C_{(p)}$ are horizontal or pseudo-horizontal. As we assumed that $C_{(p)}$ has finite Voronoi region, there must be at least one site inside each area. Otherwise, from that area the region will be open. In counterclockwise order, the intersection of the edge relevant the last site of each area, and the first edge of the next adjacent area will make four intersections which are 90° or pseudo-90° and the remaining intersections are 0° -intersections (by Lemma 2). Note that the intersection between the edge of the cell of last site in area $A^8 \cup A^1 \cup H^1$ and the first edge of $A_2 \cup A_3 \cup V_1$ occurs in area ***I*** and so on. Then there is exactly one 90° or pseudo-90° -intersection in each area ***I, II, III*** or ***IV***.

Obviously White starts any ***k-th*** round of the game by placing his ***k-th*** point, w_k, and Black terminate it by placing her ***k-th*** point, b_k, on the playing board ***Q***. The total area which is obtained by Black at the end of round k is denoted by SB_k and the total area which is obtained by White is denoted by SW_k.

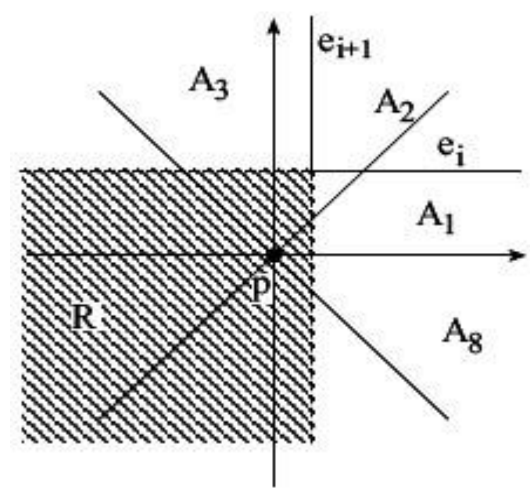

Figure 4: *Proof of Lemma 2*

3. The game

In this version of Voronoi game, we suppose that the board does not contain center of symmetry. In this version, we will see that Black has a winning strategy or they tie. In other words, we will find a strategy for Black's play such that $SB_n \geq SW_n$.

For a point A inside a symmetric polygon Q, the mirror-image of point A respect to the symmetry center of Q, O, which is denoted by A', is a point inside Q such that O is the middle point of line segment AA'.

Theorem 1: The second player, Black, never loses the game. Now for proving the above claim, we present a winning strategy for the second player. In ***k-th*** round:

If $k \leq n - 1$, Black places her point at location w'_k.

If $k = n$, Black places her point at location w0n or at a location close to w'_n with distance ε which is determined later.

Clearly if Black places her point at w0n they tie. We will show that sometimes there is a location very close to w'_n such that if Black places her point there, she can win the game. Therefore for proving the above theorem it suffices to prove this lemma.

Lemma 4: In a Manhattan-Voronoi game board, let b be a virtual black point on board, sometimes we can move b a little, horizontally or vertically, such that the area of $C_{(b)}$ increases.

From this lemma, sometimes there is a location very close to w'_n such that if Black paces her last point at there, then $|C_{(w'n)}| < |C_{(bn)}|$, i.e., Black can win the game. For proving this lemma, we need some notations.

Definition: Let P be a polygon whose sides are parallel to the x-axis, y-axis, or diagonal lines within angle л/4 or 3л/4. Let e be a side of P. If e is not horizontal or vertical, the orthogonal length of e is defined by $|e| / \sqrt{2}$, otherwise its orthogonal length is its length. The orthogonal length of a subchain of P is the sum of orthogonal lengths of line segments of chain and denoted by $OL_{(.)}$.

Let p be a site in a Manhattan-Voronoi diagram. In polygon $C_{(p)}$ draw a vertical line l through p.

Line $\boldsymbol{l}$ lies inside $\boldsymbol{C_{(P)}}$ entirely because Voronoi cells are star-shaped in Manhattan-Voronoi diagram. The border of $\boldsymbol{C_{(p)}}$ is divided into two subchains by line $\boldsymbol{l}$, the right subchain of $\boldsymbol{C_{(p)}}$ that is denoted by $\boldsymbol{R_{(p)}}$ and the left, $\boldsymbol{L_{(p)}}$. Then draw a horizontal line l0 through $\boldsymbol{p}$. Line l0 lies inside $\boldsymbol{C_{(p)}}$ entirely too. The border of $\boldsymbol{C_{(p)}}$ is divided into two subchains by line l0, the upper chain of $\boldsymbol{C_{(p)}}$ that is denoted by $\boldsymbol{U_{(p)}}$ and the lower, $\boldsymbol{D_{(p)}}$. Let p be a black site, the segments of $\boldsymbol{R_{(p)}}$ which are adjacent to a white cell are denoted by $\boldsymbol{Rw_{(p)}}$. Similarly we define $\boldsymbol{Lw_{(p)}}$, $\boldsymbol{Uw_{(p)}}$ and $\boldsymbol{Dw_{(p)}}$.

First we place a virtual black point at $\boldsymbol{w'_n}$. If $\boldsymbol{OL(R_w(w'_n)) = OL(L_w(w'_n))}$ and $\boldsymbol{OL(U_w(w'_n)) = OL(D_w(w'_n))}$, Black places $\boldsymbol{b_n}$ at $\boldsymbol{w'_n}$ and they tie. Otherwise, without less of generality assume that $\boldsymbol{OL(R_w(w'_n)) > OL(L_w(w'_n))}$, then Black places $\boldsymbol{b_n}$ at right side of $\boldsymbol{w'_n}$, on a horizontal line through w0n such that distance between $\boldsymbol{b_n}$ and $\boldsymbol{w'_n}$ is a small number $\boldsymbol{\varepsilon}$ which is determined later. In this case we will show that she can win the game.

Now we want to move virtual site $\boldsymbol{b}$ from position $\boldsymbol{w'_n}$ horizontally and compute the variation of area of is Voronoi cell as a function of coordinates of new position. Let the origin of coordinate system lies on $\boldsymbol{w'_n}$ and call the four areas counterclockwise by area ***I, II, III*** and ***IV*** and $\boldsymbol{\{ p_1, p_2, \ldots, p_m \}}$ is the neighbors of b in Manhattan-Voronoi diagram which are sorted around $\boldsymbol{b}$ conterclockwisely in order to their angle with positive direction of x-axis.

When $\boldsymbol{b=(x,y)}$, the difference between the orthogonal length of $\boldsymbol{R_w(x, y)}$ and $\boldsymbol{L_w(x, y)}$ is denoted by $\Delta \boldsymbol{P_h(x, y) = OL(R_w(x, y)) - OL(L_w(x, y))}$. Suppose that b moves from $\boldsymbol{(0, 0)}$ ***to*** $\boldsymbol{(x, 0)}$. We denote the difference between the areas of $\boldsymbol{C(0, 0)}$ and $\boldsymbol{C(x, 0)}$ by $\Delta \boldsymbol{S(x)}$. We want to put some constraints on $\boldsymbol{x}$ such that the topology of cell $\boldsymbol{C(0, 0)}$ does not change. Clearly $\boldsymbol{x}$ must be less than a real positive number that is dependent to its neighbors. We call this number β . So we have $\boldsymbol{x < \beta}$. If b moves horizontally, the new Voronoi cell is changed as shown in figure 5(a). We want to find $\Delta \boldsymbol{S(x)}$.

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ISSN= 2345-3877

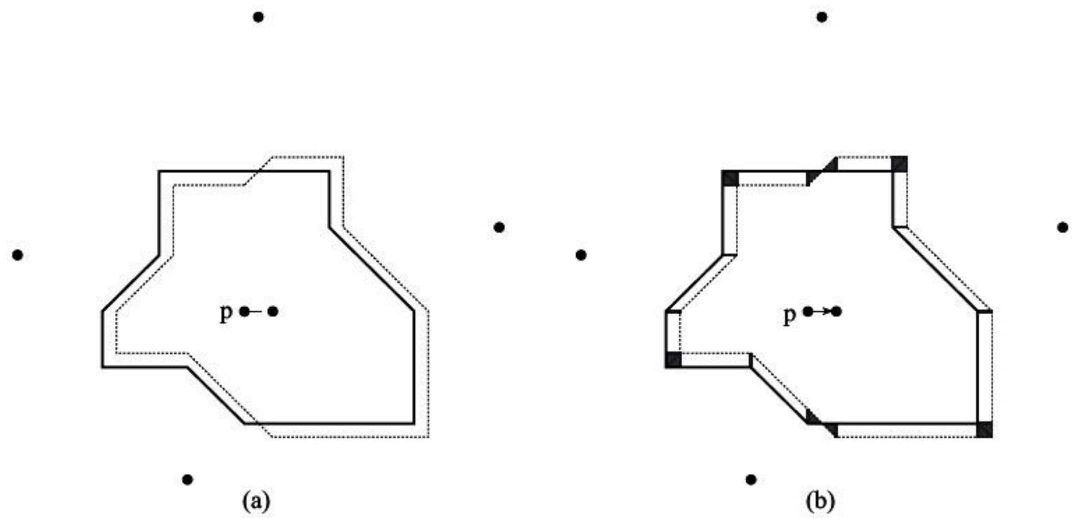

Figure 5: *(a) The new Voronoi cell, (b) The isosceles right triangles are created when we have a 90° or pseudo-90°-intersection and when $C(x,0)$ intersects $C(0,0)$.*

The bounded area between polygons $C(0, 0)$ and $C(x, 0)$ can be subdivided into some rectangles with width $x/2$ and parallelograms with height $x/(2\sqrt{2})$ and isosceles right triangles with side length $x/2$. The rectangles are erected on horizontal and vertical edges of polygon $C(0, 0)$ and parallelograms are erected on diagonal edges of polygon $C(0, 0)$. The isosceles right triangles are created when we have a 90° or pseudo- 90° -intersection and when $C(x, 0)$ intersects $C(0, 0)$. See figure 5(b). Note that 0 -intersections cannot make any triangles.

Let N_1 be the number of 90° -intersections of $C_{(b)}$ inside area I and M_1 be the number of pseudo- 90° -intersections of $C_{(b)}$ inside area I. Similarly N_i and M_i are defined where $2 \le i \le 4$. By Lemma 3, $N_i = 0$ or 1, $M_i = 0$ or 1 and $N_i + M_i = 1$ where $1 \le i \le 4$.

Theorem 2: $\Delta S(x) = (\Delta P_h(0, 0) / 2)\, x - (N/8)\, x^2$ where N is a non-negative integer number less than or equal 4.

Proof: Let S_G be the gained area by b after movement and S_L be the lost area. S_G is composed by some rectangles, parallelograms and small isosceles right triangles that rectangles and parallelograms are made by segments of $R_w(0, 0)$ and triangles are made by 90° or pseudo-90° -intersections at $R_w(0, 0)$. Also at two ends of $R_w(0, 0)$, at most six triangles are lost. So $S_G = (x/2)OL(R_w(0, 0))+n_1(1/2)(x/2)(x/2)-6(1/2)(x/2)(x/2)$. Now we want to find n_1. At every 90° -intersection two triangles are made and at every pseudo-90° -intersections on triangle is made. So $n_1 = 2(N_1 + N_4) + M_1 + M_4$. Similarly $S_L = (x/2)OL(L_w (0, 0)) - n_2(1/2)(x/2)(x/2) + 2(1/2)(x/2)(x/2)$ where $n_2 = 2(N_2 + N_3) + M_2 + M_3$. Therefore we have $\Delta S_{(x)} = S_G - S_L = (x/2)\ \Delta P_h(0,0)+(n_1 - n_2 - 8)(x^2/8)$. Let $n = 8+n_2 - n_1$, so $N = 8-(2(N1+N2+N3+N4)+M1+M2+M3+M4) = 8-(4+N1+N2+N3+N4) = 4-(N1+N2+N3+N4) \geq 0$ and $N \leq 4$. So we are done.

Note that N is only depended to the shape of $C_{(b)}$ and can be computed before movement. Also it remains x if the topology of $C_{(b)}$ does not change after movement.

If b moves from $(0, 0)$ to $(x, 0)$ that $x > 0$, parabola $\Delta S(x)$ has a global maximum at $x = (2P_h(0, 0))/N$ and $\Delta S(x) > 0$ if and only if $0 < x < (4P_h(0, 0))/N$. So if $P_h(0, 0) > 0$ we can move b to the position $(\varepsilon, 0)$ where $\varepsilon < (4P_h(0, 0))/N$ and this leads to increase the area of $C(0, 0)$ and the proof of Lemma 5 is now completed.

4. Conclusion

We considered the Manhattan-Voronoi game on a symmetric board Q, played in n rounds when first player does not place his last point at center of symmetry. First we showed that in a Manhattan Voronoi diagram a site can move a little around its position such that the topology of its cell does not change and its area increases and used it to present a winning for the second player.

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